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On the Compton Scattering in String Theory

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Abstract: We explicitly compute the Compton amplitude for the scattering of a photon and a (massless) “electron/positron” at tree level and one loop, in a four-dimensional fermionic heterotic string model. We comment on the relationship between the amplitudes we compute in string theory and the corresponding ones in field theory.

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1 Introduction

The computation of scattering amplitudes of string states through the Polyakov formula is one of the most powerful tools we have to study the properties of first-quantized (perturbative) string theories. Computations of tree-level (i.e. genus-zero) scattering amplitudes in closed string theories often appeared in the literature, mostly in the case where the external states are space-time bosonic particles. One-loop, i.e. genus-one, amplitudes have been computed in the case where the external states are space-time bosons. These computations led to many interesting results, in both string theory and field theory. Indeed one can compare directly the scattering amplitudes for the “same” external states in field theory and string theory, or one can take the field-theory limit of a string amplitude and compare it with the field theoretical one. Obviously the two expressions obtained in this way must be identical, but the way in which string theory reproduces the field theoretical amplitudes can lead to the discovery of new features of field theory (see for instance refs. [1, 2, 3]).

Very few one-loop amplitudes having space-time fermions as external states have appeared in the literature (see for example ref. [4]), mostly because of some technical issues appearing in the explicit computations of these amplitudes, as discussed for example in ref. [5].

In this paper we present one of the simplest four-point one-loop scattering amplitude, which involves external space-time fermions, that is the Compton scattering of an “electron/positron” and a photon. Here we call “electron” (or “positron”) a massless space-time fermion charged under a $U(1)$ component of the total gauge group. This state does not correspond directly to what we would usually call a, let us say, electron, since it is massless, charged under other components of the total gauge group, and also carries some (stringy) family labels. One can easily extend the results we present in this letter to the case of the scattering of a massless “quark” on a gluon, since this requires only some simple modifications of the left-moving part of our equations.

In doing these computations, we have chosen a specific “simple” four-dimensional heterotic string model. One of the particular properties of this model is that its space-time spectrum depends on a set of parameters and, as described in ref. [5], only for some values of these parameters is supersymmetric. Of course the string scattering amplitudes depend on the particular spectrum and on the details of the

compactification from ten dimensions of the chosen model. Anyway, as experience has shown, we expect that the general features of the string scattering amplitudes are independent of the specific string model chosen for the computation, in particular when one takes the field-theory limit.

We consider a fermionic four-dimensional heterotic string model in the formalism of Kawai, Lewellen and Tye [6] (see also [7, 8]). In this construction all the degrees of freedom other than those associated with the four-dimensional space-time coordinates are described by free world-sheet fermions. The fields of the model are: four space-time coordinate fields $X^\mu(z, \bar{z})$, twenty-two left-moving complex fermions $\bar{\psi}_{(\bar{l})}(\bar{z})$ (with $\bar{l} = \bar{1}, \dots, \bar{22}$), eleven right-moving complex fermions $\psi_{(l)}(z)$ (with $l = 23, \dots, 33$) of which $\psi_{(32)}$ and $\psi_{(33)}$ are the world-sheet superpartners of the space-time coordinates X_μ , right-moving superghosts β, γ , left- and right-moving reparametrization ghosts \bar{b}, \bar{c} and b, c . In ref. [5] the reader can find a complete description of the model, and further details on the conventions on the spin structures, and the vertex operators of the model under consideration can be found in refs. [9, 10]. The gauge group of the model is $SO(14) \otimes SO(14) \otimes SO(4) \otimes U(1) \otimes SO(10)$. The $U(1)$ charge, which we will be interested in, is associated with the seventeenth left-moving fermion. The vertex operator associated with the $U(1)$ gauge boson (the photon) with polarization ϵ and momentum k in the superghost picture $q = -1$ is [11]

$$\mathcal{V}_{\text{photon}}^{(-1)}(z, \bar{z}; k; \epsilon) = \frac{\kappa}{\pi} \bar{\psi}_{(\bar{17})} \bar{\psi}_{(\bar{17})}^*(\bar{z}) \epsilon \cdot \psi(z) e^{-\phi(z)} (c_{(34)})^{-1} e^{ik \cdot X(z, \bar{z})}, \quad (1)$$

where the gravitational coupling κ is related to Newton's constant by $\kappa^2 = 8\pi G_N$, $\epsilon \cdot \epsilon = 1$, $k^2 = \epsilon \cdot k = 0$ and $c_{(34)}$ is the cocycle associated to the superghosts. The picture-changed version of the same vertex is

$$\mathcal{V}_{\text{photon}}^{(0)}(z, \bar{z}; k; \epsilon) = -i \frac{\kappa}{\pi} \bar{\psi}_{(\bar{17})} \bar{\psi}_{(\bar{17})}^*(\bar{z}) [\epsilon \cdot \partial_z X(z) - ik \cdot \psi(z) \epsilon \cdot \psi(z)] e^{ik \cdot X(z, \bar{z})}. \quad (2)$$

The vertex operator for the massless fermions of momentum k , $U(1)$ charge $\pm 1/2$ and superghost charge $q = -1/2$ is given by

$$\mathcal{V}^{(-1/2)}(z, \bar{z}; k; \mathbf{V}) = \frac{\kappa}{\pi} \mathbf{V}^A S_A(z, \bar{z}) e^{-\frac{1}{2}\phi(z)} (c_{(34)})^{-1/2} e^{ik \cdot X(z, \bar{z})}, \quad (3)$$

where S_A is the spin field that creates the Ramond ground state representing the fermion from the conformal vacuum:

$$S_A = \left(\prod_{l=1}^7 \bar{S}_{\bar{a}_l}^{(\bar{l})}(\bar{z}) \right) \bar{S}_{\bar{a}_{17}}^{(\bar{17})}(\bar{z}) \left(\prod_{l=24, 28, 29} S_{a_l}^{(l)}(z) \right) S_{\alpha}^{(l)}(z) \quad (4)$$

and the “spinor” \mathbf{V}^A decomposes accordingly. All indices (\bar{a}_l, a_l) take values $\pm 1/2$: $(\bar{a}_1, \dots, \bar{a}_7)$ are indices of the first $SO(14)$; $\bar{a}_{17} = \pm 1/2$ is the $U(1)$ charge; $\alpha = (a_{32}, a_{33})$ is the four-dimensional space-time spinor index, while (a_{24}, a_{28}, a_{29}) are just enumerative family indices. The explicit expression of the spin fields is obtained by bosonizing the world-sheet fermions, see refs. [11, 5, 9]. We have introduced the dimensionless momentum $k_\mu = \sqrt{\frac{\alpha'}{2}} p_\mu$ and the Minkowski space-time metric is $\eta_{\mu\nu} = (-1, 1, 1, 1)$.

2 The tree-level amplitude

We start by discussing the tree-level scattering and comparing the string-theory amplitude with the field-theory one. The tree-level, i.e. genus-zero, string Compton scattering amplitude turns out to be practically independent of the chosen string model, as will be obvious from eq. (6).

We adopt the notations of ref. [10] for the general formula expressing the connected part of a string scattering amplitude as a two-dimensional correlation function of vertex operators. At tree level, the amplitude for the scattering of two photons and two massless fermions is then written as

$$\begin{aligned}
T_{g=0} (e^\pm + \gamma \rightarrow e^\pm + \gamma) = \\
C_{g=0} \int d^2 z_1 \langle \Pi(w) \mathcal{V}_{\text{photon}}^{(-1)}(z_1, \bar{z}_1; k_1; \epsilon_1) \bar{c}(\bar{z}_2) c(z_2) \mathcal{V}_{\text{photon}}^{(-1)}(z_2, \bar{z}_2; k_2; \epsilon_2) \\
\bar{c}(\bar{z}_3) c(z_3) \mathcal{V}^{(-1/2)}(z_3, \bar{z}_3; k_3; \mathbf{W}_3) \bar{c}(\bar{z}_4) c(z_4) \mathcal{V}^{(-1/2)}(z_4, \bar{z}_4; k_4; \mathbf{V}_4) \rangle, \quad (5)
\end{aligned}$$

where $C_{g=0} = -4\pi^3/(\kappa^2 \alpha')$ is a constant giving the proper normalization of the vacuum amplitude [12, 10], \mathbf{V}_4 is the “spinor” of the incoming electron, while the vector \mathbf{W}_3 is associated with the outgoing one as in eq. (8.11) of ref. [10], see also below.

At tree level one generically takes all space-time fermionic vertex operators to have the superghost charge $q = -1/2$ and all the bosonic ones to have the superghost charge $q = -1$. We denote by $\Pi(w)$ the Picture-Changing Operator (PCO) needed to compensate the superghost vacuum charge [13, 11]. The amplitude does not depend on the PCO insertion point w , even if, in general, it is not so straightforward to show such a property on the final form of the amplitude. In the case at hand, however, it is easy to show that in the final form of the amplitude the dependence on the PCO insertion point w cancels explicitly. Of course, one can always take the limit

$w \rightarrow z_1$ (or $w \rightarrow z_2$) in eq. (5), eliminating the PCO operator from the beginning and transforming one photon vertex operator into its picture-changed version with superghost charge $q = 0$. Technically it is often convenient to keep the PCO at a generic point on the world-sheet, since the independence of the final result from it is a valuable check on the correctness of the computation.

After standard manipulations, the final form of the tree amplitude is

$$\begin{aligned}
T_{g=0} (e^\pm + \gamma \rightarrow e^\pm + \gamma) = & \\
& - \frac{\kappa^2}{(\alpha')^{3/2}} \left[\left(\frac{1}{s} + \frac{1}{u} \right) \mathbf{W}_3^T \not{p}_1 \not{\epsilon}_1 \not{\epsilon}_2 \mathbf{C} \mathbf{V}_4 - \left(\frac{2}{s} \right) (\epsilon_1 \epsilon_2) \mathbf{W}_3^T \not{p}_1 \mathbf{C} \mathbf{V}_4 \right. \\
& \quad \left. + \left(\frac{2}{s} \right) (\epsilon_2 p_1) \mathbf{W}_3^T \not{\epsilon}_1 \mathbf{C} \mathbf{V}_4 + \left(\frac{2}{s} (\epsilon_1 p_4) - \frac{2}{u} (\epsilon_1 p_3) \right) \mathbf{W}_3^T \not{\epsilon}_2 \mathbf{C} \mathbf{V}_4 \right] \\
& \times \left[1 - \frac{\alpha' s u}{t(+\frac{1}{4}\alpha' t)} \right] \frac{\Gamma(1 - \frac{1}{4}\alpha' s) \Gamma(1 - \frac{1}{4}\alpha' t) \Gamma(1 - \frac{1}{4}\alpha' u)}{\Gamma(1 + \frac{1}{4}\alpha' s) \Gamma(1 + \frac{1}{4}\alpha' t) \Gamma(1 + \frac{1}{4}\alpha' u)}, \tag{6}
\end{aligned}$$

where the Mandelstam variables s, t, u are defined as: $s = -(p_1 + p_4)^2 = -(p_2 + p_3)^2$, $t = -(p_1 + p_2)^2 = -(p_3 + p_4)^2$, $u = -(p_2 + p_4)^2 = -(p_1 + p_3)^2$.

The “spinors” \mathbf{V} are normalized according to ref. [10], i.e. $\mathbf{V}^\dagger(p) \mathbf{V}(p) = \sqrt{\alpha'} |p_0|$, so that there is a factor $(\alpha')^{1/4} / \sqrt{2}$ with respect to the normalization of spinors in field theory,¹ and the vector \mathbf{W}_3^T is related to the “spinor” \mathbf{V}_3^T , describing the outgoing electron, by $\mathbf{W}_3^T = \mathbf{V}_3^T \sigma_1^{(34)} \Sigma$ (see ref. [10] for more details). Finally, the gauge coupling constant is expressed in terms of the constant κ by the relation $e^2 = \kappa^2 / (2\alpha')$.

After these substitutions have been performed, the string amplitude can be easily compared with the field theoretical one. It is interesting to note that, after expressing eq. (6) in terms of the field-theory variables, the dependence on α' is limited to the last line of eq. (6). Indeed the first two lines of eq. (6) give exactly the tree-level field theory result for the Compton amplitude (see for instance [15, 14]):

$$\begin{aligned}
T(e^\pm + \gamma \rightarrow e^\pm + \gamma) = & -e^2 \left[\left(\frac{1}{s} + \frac{1}{u} \right) \bar{\mathbf{u}}(p_3) \not{p}_1 \not{\epsilon}_1 \not{\epsilon}_2 \mathbf{u}(p_4) - \left(\frac{2}{s} \right) (\epsilon_1 \epsilon_2) \bar{\mathbf{u}}(p_3) \not{p}_1 \mathbf{u}(p_4) \right. \\
& \quad \left. + \left(\frac{2}{s} \right) (\epsilon_2 p_1) \bar{\mathbf{u}}(p_3) \not{\epsilon}_1 \mathbf{u}(p_4) + \left(\frac{2}{s} (\epsilon_1 p_4) - \frac{2}{u} (\epsilon_1 p_3) \right) \bar{\mathbf{u}}(p_3) \not{\epsilon}_2 \mathbf{u}(p_4) \right] \tag{7}
\end{aligned}$$

and in the $\alpha' \rightarrow 0$ limit the last line of eq. (6) goes exactly to +1.

Thus, at tree level, the string Compton amplitude has the interesting feature that it factorizes in a part that coincides with the field-theory amplitude and a part that

¹ For our conventions in field theory and the normalization of spinors we follow, for instance, ref. [14], with the opposite Minkowski metric.

gives the string-theory corrections to the scattering. The stringy correction is in the usual Veneziano form, showing the appearance of the poles corresponding to all possible intermediate string states.

3 The one-loop amplitude

In this section we present the one-loop, i.e. genus-one, Compton scattering amplitude in our four-dimensional string model. We have chosen to compute the scattering of a chiral massless fermion on a photon since only string massless states survive in the field-theory limit, which allows us to make easy comparisons with the field-theory results.

The presence of massless external states implies that our amplitude suffers from infrared divergences which are the same as the ones appearing in field theory and are due to the emission/absorption of soft photons/electrons. The easiest way to deal with these divergences is to introduce an appropriate infrared cut-off.

The string scattering amplitude is free from the ultraviolet divergences that appear in field theory and it automatically regulates also the chiral massless fermions describing our electrons. These divergences of course reappear in the field-theory limit basically as divergences in $\alpha' \rightarrow 0$. The string scattering amplitude is not anyway free of divergences; instead, as is well known, there appear divergences in the integrations over the moduli. The physical interpretation of these divergences has been discussed for example in refs. [10], [16]—[22], and is related to the unitarity of the scattering amplitude. The amplitude that we present below is unregulated, which means formally Hermitian, and to obtain the amplitude with the poles and cuts required by unitarity one has to apply a procedure of analytic continuation (the stringy version of the Feynman “ $+i\epsilon$ ”), similar for example to one of those described in refs. [19, 20, 21].

Keeping these points in mind, we could try to compare the string scattering amplitude that we will present in this section, with the corresponding one in field theory, as we did for the tree-level computation in the previous section. In this case, the integrals over the moduli in the string amplitude correspond to the integrals over the Schwinger parameters in field theory (see for example ref. [2]). At a first look,²

² We do not display the field-theory expression of the scattering amplitude at one loop, but the reader can easily derive it her/himself.

one immediately notices that the string theory expression we give below contains more kinematical terms than the one in field theory. Of course, one should compare the string- and field-theory expressions only after having done the integrals over the Schwinger parameters in field theory, and the integrals over the moduli and the sum over the spin-structures in string theory. In field theory the integrals can be done, but in string theory the integrals and the sums usually cannot. This prevents us from making any particular claim about the properties of the string-theory amplitude before making a suitable approximation or taking the field-theory limit.

The starting point of the computation of the one-loop Compton scattering amplitude in string theory is given by the connected part of the S matrix:

$$\begin{aligned}
T_{g=1} (e^\pm + \gamma \rightarrow e^\pm + \gamma) = & \\
C_{g=1} \int d^2\tau d^2z_2 d^2z_3 d^2z_4 \sum_{m_i, n_j} C_\beta^\alpha \langle \langle (\eta_\tau|b)(\eta_{z_2}|b)(\eta_{z_3}|b)(\eta_{z_4}|b) & \\
\times c(z_1)c(z_2)c(z_3)c(z_4)|^2 \Pi(w) \mathcal{V}_{\text{photon}}^{(0)}(z_1, \bar{z}_1; k_1; \epsilon_1) \mathcal{V}_{\text{photon}}^{(0)}(z_2, \bar{z}_2; k_2; \epsilon_2) & \\
\times \mathcal{V}^{(-1/2)}(z_3, \bar{z}_3; k_3; \mathbf{W}_3) \mathcal{V}^{(-1/2)}(z_4, \bar{z}_4; k_4; \mathbf{V}_4) \rangle \rangle, & \quad (8)
\end{aligned}$$

where $C_{g=1} = 1/(2\pi\alpha')^2$ [12, 10]. In higher-genus computations, it is convenient [5, 10] to use bosonic vertex operators in the zero super ghost picture and fermionic vertex operators in the $-1/2$ picture. In our case this leaves only one PCO at an arbitrary point w . As a consistency check we have verified that the final expression of the amplitude, eq. (9), is independent of w . Notice that, as was already the case in ref. [5], it is not possible to explicitly eliminate w from eq. (9), as we did at tree level, but since this equation does not depend on w , one can fix w to the most convenient value.

The technical details of the computation of eq. (8), which has been done following the procedure of ref. [5], will be discussed elsewhere. Let us only note that the main difficulties come from the presence of the spin fields. One way of computing these correlators is to bosonize all world-sheet fermions [11, 23]. At this point the computation of the correlators becomes almost trivial, but Lorentz covariance is lost and can be recovered only after some technical steps that involve, among other things, also the use of some non-trivial identities in theta functions, mostly of the form of the trisecantic identity [24].

The final form of the one-loop amplitude is

$$T_{g=1}(e^\pm + \gamma \rightarrow e^\pm + \gamma) =$$

$$\begin{aligned}
& \frac{e^4}{\pi^6} \sum_{m_i, n_j} C_\beta^\alpha e^{2\pi i K_{GSO}} \int \frac{d^2 \tau}{(\text{Im} \tau)^2} (\bar{\eta}(\bar{\tau}))^{-24} (\eta(\tau))^{-12} \\
& \times \int d^2 z_2 d^2 z_3 d^2 z_4 \frac{\sqrt{\omega(z_3) \omega(z_4)}}{\bar{\omega}(\bar{z}_1) \omega(z_1) \omega(w)} \exp \left[\sum_{i < j} (k_i k_j) G_B(z_i, z_j) \right] \\
& \times \mathcal{T}_L \left[\frac{\bar{\alpha}}{\beta} \right] (z_1, z_2, z_3, z_4, w) \times \mathcal{T}_R \left[\frac{\alpha}{\beta} \right] (z_1, z_2, z_3, z_4, w) , \tag{9}
\end{aligned}$$

where $\omega(z) = 1/z$ and

$$\begin{aligned}
\mathcal{T}_L (z_1, z_2, z_3, z_4, w) = & \prod_{l=1-7, 17} \bar{\Theta} \left[\frac{\bar{\alpha}_l}{\beta_l} \right] \left(\frac{1}{2} \bar{\nu}_{34} | \bar{\tau} \right) \times \prod_{l=8}^{16} \prod_{l=18}^{22} \bar{\Theta} \left[\frac{\bar{\alpha}_l}{\beta_l} \right] (0 | \bar{\tau}) \times (\bar{E}(\bar{z}_3, \bar{z}_4))^{-2} \\
& \times \left\{ \partial_{\bar{z}_1} \partial_{\bar{z}_2} \log \bar{E}(\bar{z}_1, \bar{z}_2) + \frac{1}{4} \partial_{\bar{z}_1} \log \frac{\bar{E}(\bar{z}_1, \bar{z}_3)}{\bar{E}(\bar{z}_1, \bar{z}_4)} \partial_{\bar{z}_2} \log \frac{\bar{E}(\bar{z}_2, \bar{z}_3)}{\bar{E}(\bar{z}_2, \bar{z}_4)} \right. \\
& + \frac{1}{2} \frac{\bar{\omega}(\bar{z}_1)}{2\pi i} \partial_{\bar{z}_2} \log \frac{\bar{E}(\bar{z}_2, \bar{z}_3)}{\bar{E}(\bar{z}_2, \bar{z}_4)} \partial_\nu \log \bar{\Theta} \left[\frac{\bar{\alpha}_{17}}{\beta_{17}} \right] (\nu | \bar{\tau})|_{\nu=\frac{1}{2}\bar{\nu}_{34}} \\
& + \frac{1}{2} \frac{\bar{\omega}(\bar{z}_2)}{2\pi i} \partial_{\bar{z}_1} \log \frac{\bar{E}(\bar{z}_1, \bar{z}_3)}{\bar{E}(\bar{z}_1, \bar{z}_4)} \partial_\nu \log \bar{\Theta} \left[\frac{\bar{\alpha}_{17}}{\beta_{17}} \right] (\nu | \bar{\tau})|_{\nu=\frac{1}{2}\bar{\nu}_{34}} \\
& \left. + \frac{\bar{\omega}(\bar{z}_1)}{2\pi i} \frac{\bar{\omega}(\bar{z}_2)}{2\pi i} \left(\bar{\Theta} \left[\frac{\bar{\alpha}_{17}}{\beta_{17}} \right] \left(\frac{1}{2} \bar{\nu}_{34} | \bar{\tau} \right) \right)^{-1} \partial_\nu^2 \bar{\Theta}(\nu | \bar{\tau})|_{\nu=\frac{1}{2}\bar{\nu}_{34}} \right\} , \tag{10}
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_R (z_1, z_2, z_3, z_4, w) = & \prod_{l=24, 28, 29, 32} \Theta \left[\frac{\alpha_l}{\beta_l} \right] \left(\frac{1}{2} \nu_{34} | \tau \right) \prod_{l=23, 25, 26, 27, 30, 31} \Theta \left[\frac{\alpha_l}{\beta_l} \right] (0 | \tau) \\
& \times \frac{(-1)^{S_{33}}}{\sqrt{2}} (E(z_3, z_4))^{-1} \left\{ \mathbf{W}_3^T h_1 \not{e}_1 \not{e}_2 \left(\Gamma^5 \right)^{\tilde{S}} \mathbf{CV}_4 \mathcal{A}_1(z_1, z_2, z_3, z_4, w) + \right. \\
& + \mathbf{W}_3^T h_1 \left(\Gamma^5 \right)^{\tilde{S}} \mathbf{CV}_4 \mathcal{A}_2(z_1, z_2, z_3, z_4, w) + \\
& + \mathbf{W}_3^T \not{e}_1 \left(\Gamma^5 \right)^{\tilde{S}} \mathbf{CV}_4 \mathcal{A}_3(z_1, z_2, z_3, z_4, w) + \\
& \left. + \mathbf{W}_3^T \not{e}_2 \left(\Gamma^5 \right)^{\tilde{S}} \mathbf{CV}_4 \mathcal{A}_4(z_1, z_2, z_3, z_4, w) \right\} , \tag{11}
\end{aligned}$$

with $\nu_i = \frac{1}{2\pi i} \log z_i$ and $\nu_{ij} = \nu_i - \nu_j$. In eq. (9), K_{GSO} is a phase factor that depends on the variables k_{ij} parametrizing the GSO projection:

$$\begin{aligned}
K_{GSO} = & (k_{02} + k_{12} + k_{14} + k_{23} + k_{24} + k_{34} + 1/2) S_1 \\
& + (k_{00} + k_{01} + k_{02} + k_{03} + k_{04} + k_{12} + k_{23} + k_{24} + 1/2) S_{17} \\
& + (k_{00} + k_{01} + k_{03} + k_{04} + k_{13} + k_{34} + 1/2) S_{24} \\
& + (k_{00} + k_{01} + k_{03} + k_{04} + k_{13} + k_{14} + 1/2) S_{29} , \tag{12}
\end{aligned}$$

$S_i = (1 - 2\alpha_i)(1 + 2\beta_i)$ and $\tilde{S} = S_{17} + S_{24} + S_{29} + S_{33}$. The coefficients \mathcal{A}_i are functions of the external momenta, polarization vectors and world-sheet coordinates:³

$$\begin{aligned}\mathcal{A}_1(z_1, z_2, w) &= (k_1 k_2) [\partial_w G_B(w, z_1) - \partial_w G_B(w, z_2)] G^+(z_1, z_2) G^-(z_1, z_2) \\ &+ \frac{1}{2} \sum_{j=1}^4 (k_1 k_j) \partial_{z_1} G_B(z_j, z_1) \partial_w G_B(z_1, w) I(z_2) + \\ &+ \frac{1}{2} \sum_{j=1}^4 (k_2 k_j) \partial_{z_2} G_B(z_j, z_2) \partial_w G_B(z_2, w) I(z_1) + \\ &+ \sum_{j=1}^4 \partial_w G_B(w, z_j) \left[(k_1 k_j) \mathcal{I}(z_2) G^+(z_1, w) + (k_2 k_j) \mathcal{I}(z_1) G^+(z_2, w) \right] \frac{G^-(z_1, z_2)}{\mathcal{I}(w)},\end{aligned}$$

$$\begin{aligned}\mathcal{A}_2(z_1, z_2, w) &= [\partial_w G_B(z_1, w) - \partial_w G_B(z_2, w)] [(\epsilon_1 \epsilon_2) \partial_{z_1} \partial_{z_2} G_B(z_1, z_2) + \\ &+ \sum_{j,i=1}^4 (\epsilon_1 k_i) (\epsilon_2 k_j) \partial_{z_1} G_B(z_i, z_1) \partial_{z_2} G_B(z_j, z_2)] + \\ &+ (\epsilon_1 k_3) \sum_{j=1}^4 (\epsilon_2 k_j) \partial_{z_2} G_B(z_j, z_2) I(z_1) [\partial_w G_B(z_2, w) - \partial_w G_B(z_3, w)] + \\ &- (\epsilon_2 k_3) \sum_{j=1}^4 (\epsilon_1 k_j) \partial_{z_1} G_B(z_j, z_1) I(z_2) [\partial_w G_B(z_1, w) - \partial_w G_B(z_3, w)] + \\ &- (\epsilon_1 \epsilon_2) \sum_{j=1}^4 \left[(k_2 k_j) \partial_{z_2} G_B(z_j, z_2) \partial_w G_B(z_2, w) B^+(z_1, w) + \right. \\ &\quad \left. - (k_1 k_j) \partial_{z_1} G_B(z_j, z_1) \partial_w G_B(z_1, w) B^-(z_2, w) \right] + \\ &- \sum_{j,i=1}^4 (\epsilon_1 k_i) (\epsilon_2 k_j) \left[\partial_{z_1} G_B(z_i, z_1) \partial_w G_B(z_j, w) B^+(z_2, w) + \right. \\ &\quad \left. - \partial_{z_2} G_B(z_j, z_2) \partial_w G_B(z_i, w) B^+(z_1, w) \right] + \\ &+ \sum_{j=1}^4 \partial_w G_B(z_j, w) \{ [(\epsilon_1 k_2) (\epsilon_2 k_j) - (\epsilon_1 \epsilon_2) (k_2 k_j)] C_2(z_1, z_2, w) + \\ &\quad + [(\epsilon_2 k_1) (\epsilon_1 k_j) - (\epsilon_1 \epsilon_2) (k_1 k_j)] C_1(z_1, z_2, w) \} + \\ &+ 2 \sum_{j=1}^4 \partial_w G_B(z_j, w) [(\epsilon_1 k_3) (\epsilon_2 k_j) D_2(z_1, z_2, w) + (\epsilon_2 k_4) (\epsilon_1 k_j) D_1(z_1, z_2, w)] + \\ &+ [(\epsilon_1 k_2) (\epsilon_2 k_1) - (\epsilon_1 \epsilon_2) (k_1 k_2)] [\partial_w G_B(z_1, w) - \partial_w G_B(z_2, w)] \times \\ &\quad \times [G^+(z_1, z_2)^2 + 2G^+(z_1, z_2) G^-(z_1, z_2)] +\end{aligned}$$

³ Even if not explicitly written, all functions listed below depend also on the world-sheet coordinates z_3, z_4 . For our conventions on the theta functions and the prime form, see ref. [5].

$$\begin{aligned}
& + 2 G^+(z_1, z_2) G^-(z_1, z_2) \{ (\epsilon_1 k_2) \sum_{j=1,3} (\epsilon_2 k_j) [\partial_w G_B(z_2, w) - \partial_w G_B(z_j, w)] + \\
& + [(\epsilon_1 \epsilon_2)(k_1 k_2) + (\epsilon_1 k_3)(\epsilon_2 k_1)] [\partial_w G_B(z_1, w) - \partial_w G_B(z_3, w)] \},
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_3(z_1, z_2, w) &= \partial_w G_B(z_1, w) [(\epsilon_2 k_1) \partial_{z_1} G_B(z_1, z_2) \sum_{j=1}^4 (k_j k_2) \partial_{z_2} G_B(z_2, z_j) + \\
& - \sum_{j,i=1}^4 (k_1 k_i) (\epsilon_2 k_j) \partial_{z_1} G_B(z_i, z_1) \partial_{z_2} G_B(z_j, z_2)] + \\
& - (k_1 k_3) \sum_{j=1}^4 (\epsilon_2 k_j) \partial_{z_2} G_B(z_j, z_2) [\partial_w G_B(z_2, w) - \partial_w G_B(z_3, w)] I(z_1) + \\
& - (\epsilon_2 k_4) \partial_w G_B(z_1, w) \sum_{j=1}^4 (k_1 k_j) \partial_{z_1} G_B(z_j, z_1) I(z_2) + \\
& - \sum_{j=1}^4 [(k_1 k_2) (\epsilon_2 k_j) - (\epsilon_2 k_1) (k_2 k_j)] \partial_w G_B(z_j, w) C_2(z_1, z_2, w) + \\
& + B^+(z_1, w) \sum_{j=1}^4 \partial_{z_2} G_B(z_j, z_2) [(k_2 k_j) (\epsilon_2 k_1) \partial_w G_B(z_2, w) + \\
& - \sum_{i=1}^4 (\epsilon_2 k_j) (k_1 k_i) \partial_w G_B(z_i, w)] + \\
& + 2 G^+(z_1, z_2) G^-(z_1, z_2) \{ (\epsilon_2 k_1) (k_1 k_3) [\partial_w G_B(z_3, w) - \partial_w G_B(z_1, w)] + \\
& - (\epsilon_2 k_4) (k_1 k_2) [\partial_w G_B(z_3, w) - \partial_w G_B(z_2, w)] \} + \\
& - 2 \sum_{j=1}^4 \partial_w G_B(z_j, w) [(\epsilon_2 k_4) (k_1 k_j) D_1(z_1, z_2, w) + (k_1 k_3) (\epsilon_2 k_j) D_2(z_1, z_2, w)],
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_4(z_1, z_2, w) &= \partial_w G_B(z_2, w) [(\epsilon_1 k_2) \partial_{z_2} G_B(z_1, z_2) \sum_{j=1}^4 (k_j k_1) \partial_{z_1} G_B(z_1, z_j) + \\
& - \sum_{j,i=1}^4 (k_2 k_i) (\epsilon_1 k_j) \partial_{z_1} G_B(z_j, z_1) \partial_{z_2} G_B(z_i, z_2)] + \\
& - (k_2 k_3) \sum_{j=1}^4 (\epsilon_1 k_j) \partial_{z_1} G_B(z_j, z_1) [\partial_w G_B(z_1, w) - \partial_w G_B(z_3, w)] I(z_2) + \\
& + (\epsilon_1 k_4) \partial_w G_B(z_1, w) \sum_{j=1}^4 (k_1 k_j) \partial_{z_1} G_B(z_j, z_1) I(z_2) + \\
& + \sum_{j=1}^4 [(k_1 k_2) (\epsilon_1 k_j) - (\epsilon_1 k_2) (k_1 k_j)] \partial_w G_B(z_j, w) C_1(z_1, z_2, w) +
\end{aligned}$$

$$\begin{aligned}
& -B^+(z_2, w) \sum_{j=1}^4 \partial_{z_1} G_B(z_j, z_1) \left[\sum_{i=1}^4 (\epsilon_1 k_j)(k_2 k_i) \partial_w G_B(z_i, w) + \right. \\
& \quad \left. - (k_1 k_j)(\epsilon_1 k_2) \partial_w G_B(z_1, w) \right] \\
& + 2G^+(z_1, z_2)G^-(z_1, z_2) [(\epsilon_1 k_3)(k_1 k_2) - (\epsilon_1 k_2)(k_1 k_3)] \times \\
& \quad \times [\partial_w G_B(z_3, w) - \partial_w G_B(z_2, w)] + \\
& + 2 \sum_{j=1}^4 \partial_w G_B(z_j, w) [(\epsilon_1 k_j)(k_1 k_3) - (k_1 k_j)(\epsilon_1 k_3)] D_1(z_1, z_2, w),
\end{aligned}$$

and finally

$$\begin{aligned}
G_B(x, \bar{x}; y, \bar{y}) &= 2 \left[\log |E(x, y)| - \frac{1}{2} \operatorname{Re} \left(\int_y^x \omega \right)^2 \frac{1}{2\pi \operatorname{Im} \tau} \right], \\
\mathcal{I} \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (z) &= \sqrt{\frac{E(z_3, z_4)}{E(z, z_3)E(z, z_4)}} \frac{\Theta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (\nu_z - \frac{1}{2}\nu_3 - \frac{1}{2}\nu_4 | \tau)}{\Theta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (\frac{1}{2}\nu_{34} | \tau)}, \\
I \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (z) &= \partial_z \log \frac{E(z, z_3)}{E(z, z_4)} + 2 \frac{\omega(z)}{2\pi i} \partial_\nu \log \Theta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (\nu | \tau) \Big|_{\nu=\frac{1}{2}\nu_{34}}, \\
G^\pm \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (z, w) &= \frac{1}{2E(z, w)} \left\{ \frac{\Theta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (\nu_{zw} + \frac{1}{2}\nu_{34} | \tau)}{\Theta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (\frac{1}{2}\nu_{34} | \tau)} \sqrt{\frac{E(z, z_3)E(w, z_4)}{E(w, z_3)E(z, z_4)}} \right. \\
& \quad \left. \pm \frac{\Theta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (\nu_{wz} + \frac{1}{2}\nu_{34} | \tau)}{\Theta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (\frac{1}{2}\nu_{34} | \tau)} \sqrt{\frac{E(w, z_3)E(z, z_4)}{E(z, z_3)E(w, z_4)}} \right\}, \\
B^\pm \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (z_i, w) &= \frac{\mathcal{I}(z_i)}{\mathcal{I}(w)} \left[G^+(z_i, w) \pm G^-(z_i, w) \right], \\
D_{1,2} \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (z_1, z_2, w) &= \frac{\mathcal{I}(z_{2,1})}{\mathcal{I}(w)} G^+(z_{1,2}, w) G^-(z_1, z_2), \\
C_{1,2} \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (z_1, z_2, w) &= \left[G^+(z_1, z_2) G^-(z_1, z_2) + \right. \\
& \quad \left. + \frac{\mathcal{I}(z_{2,1})}{\mathcal{I}(w)} G^+(z_{1,2}, w) \left(G^+(z_1, z_2) + G^-(z_1, z_2) \right) \right].
\end{aligned}$$

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References

- [1] V.S. Kaplunovsky, Nucl.Phys. **B307** (1988) 145 [hep-th/9205070].
- [2] Z. Bern and D. Kosower, Nucl.Phys. **B379** (1992) 451.
- [3] P. Di Vecchia, A. Lerda, L. Magnea, R. Marotta and R. Russo, Nucl.Phys. **B469** (1996) 235 [hep-th/9601143].
- [4] J.J. Atick, L.J. Dixon and A. Sen, Nucl.Phys. **B292** (1987) 109;
J.J. Atick and A. Sen, Phys.Lett. **186B** (1987) 339, Nucl.Phys. **B286** (1987) 189 and **B293** (1987) 317.
- [5] A. Pasquinucci and K. Roland, Nucl.Phys. **B440** (1995) 441 [hep-th/9411015].
- [6] H. Kawai, D.C. Lewellen and S.H.H.Tye, Nucl.Phys. **B288** (1987) 1.
- [7] I. Antoniadis, C. Bachas, C. Kounnas and P. Windey, Phys.Lett. **171B** (1986) 51;
I. Antoniadis, C. Bachas and C. Kounnas, Nucl.Phys. **B289** (1987) 87;
I. Antoniadis and C. Bachas, Nucl.Phys. **B298** (1988) 586.
- [8] R. Bluhm, L. Dolan and P. Goddard, Nucl.Phys. **B309** (1988) 330.
- [9] A. Pasquinucci and K. Roland, Phys.Lett. **B351** (1995) 131 [hep-th/9503040].
- [10] A. Pasquinucci and K. Roland, Nucl.Phys. **B457** (1995) 27 [hep-th/9508135].
- [11] D. Friedan, E. Martinec and S. Shenker, Nucl.Phys. **B271** (1986) 93.
- [12] G. Cristofano, R. Marotta and K. Roland, Nucl.Phys. **B392** (1993) 345.
- [13] For a review, see E. D'Hoker and D.H. Phong, Rev.Mod.Phys. **60** (1988) 917.
- [14] M. Le Bellac, “*Des phénomènes critiques aux champs de jauge*”, InterEditions et Editions du CNRS, Paris, 1988.
- [15] L.M. Brown and R.P. Feynman, Phys.Rev. **85** (1952) 231.
- [16] S. Weinberg, “*Radiative corrections in string theory*”, Talk given at the APS meeting 1985, in DPF Conf. 1985, p. 850.

- [17] W. Fischler and L. Susskind, Phys.Lett. **B171** (1986) 383, and **B173** (1986) 262.
- [18] A.A. Tseytlin, in “*Trieste Superstrings 1989*”, 487, and references therein.
- [19] K. Aoki, E. D’Hoker and D.H. Phong, Nucl.Phys. **B342** (1990) 149;
 E. D’Hoker and D.H. Phong, Phys.Rev.Lett. **70** (1993) 3692, Theor.Math.Phys. **98** (1994) 306 [hep-th/9404128], Nucl.Phys. **B440** (1995) 24 [hep-th/9410152].
- [20] A. Berera, Nucl.Phys. **B411** (1994) 157.
- [21] J.L. Montag and W.I. Weisberger, Nucl.Phys. **B363** (1991) 527.
- [22] A. Pasquinucci and K. Roland, Nucl.Phys. **B485** (1997) 241 [hep-th/9608022].
- [23] V.A. Kostelecky, O. Lechtenfeld, W. Lerche, S. Samuel and S. Watamura, Nucl.Phys. **B288** (1987) 173.
- [24] J. Fay, “*Theta functions on Riemann Surfaces*”, Lecture Notes in Mathematics **352**, Springer-Verlag, Berlin, 1973;
 D. Mumford, “*Tata Lectures on Theta 1*”, Progress in Mathematics **28**, Birkhäuser, Boston, 1983;
 J. Fay, Duke Math. J. **51** (1984) 109;
 J. Fay, Proceedings of Symposia in Pure Mathematics **49** (1989) 485.